4. Moment Of Inertia

Moment of Inertia:

A quantity expressing a body (or) bodies tendency to resist angular acceleration. "It is the sum of the product of the mass of each particle in the body with the square of its distance from the axis of rotation". The moment of inertia depends on how mass is distributed around on axis of rotation and will vary depending on the chosen axis.

Area moment of inertia:

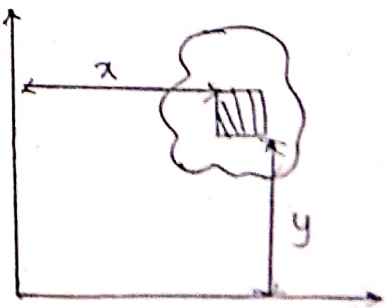
The moment of inertia of an area is called the area moment of inertia (or) the second moment of area.

Mass moment of inertia:

The moment of inertia of a mass of a body is called the mass moment of inertia.

Moment of inertia of an Area:

Consider a lamina of area "A" in the region bounded by two reference axes 'ox' and 'oy' as shown in figure.



It consists of a number of small elemental area such that the sum of these elemental area is equal to the total area 'A'

Let dA = Area of any element situated at a distance of 'x' and 'y' from the axes shown in figure.

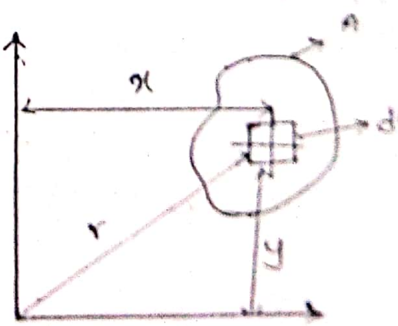
The moment of inertia of area 'A' with respect to x axis and y axis which is also called the second moment of area and is defined by $\Rightarrow I_{xx} = \int dA \times y^2$

$$\Rightarrow I_{yy} = \int dA \times x^2$$

Perpendicular axis theorem:

"It states that the moment of inertia of an area with respect to an axis perpendicular to the x-axis and y-axis and passing through origin will be equal to the sum of moment of inertia of the same area about "x-axis" and "y-axis". It is also known as "polar moment of inertia".

Proof:



→ Consider an elemental area "dA" at a distance of "x" and "y" from the axes "ox" & "oy" respectively

→ Let "oz" be the perpendicular axis to the 'ox' and 'oy' axis passing through origin "o".

r = distance of centre of gravity of elemental area from "oz" axis

$$\therefore \quad I_{xx} = \int dA \times y^2$$

$$I_{yy} = \int dA \times x^2$$

$$I_{zz} = \int dA \times r^2$$

from the statement

$$I_{zz} = I_{xx} + I_{yy}$$

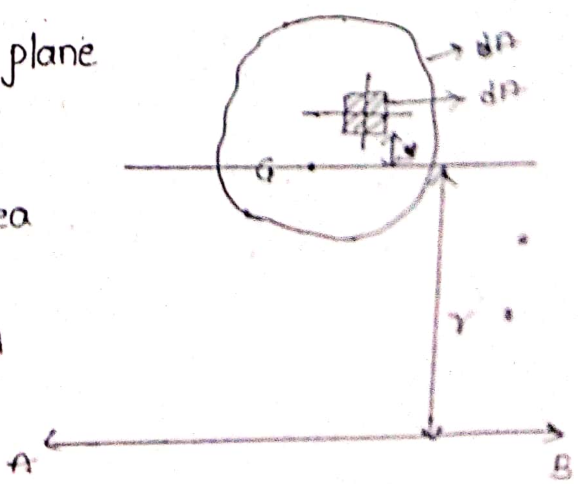
$$\int dA r^2 = \int dA \times y^2 + \int dA \times x^2$$

$$\therefore \boxed{r^2 = x^2 + y^2}$$

Parallel axis theorem

"It states that the moment of inertia of a plane area with respect to any reference axis in its plane is equal to the sum of moment of inertia with respect to a parallel centroidal axis and the product of the total area and the square of the distance between the two axes."

Let "A" be the area of the plane lamina as shown in figure.



Consider a small elemental area "dA" at a distance "y" from the axis passing through the centroid of the area "A"

Let "r" be the distance between the parallel centroidal axis and the reference axis AB.

∴ Moment of inertia of the elemental area "dA" about the reference axis "AB" is

$$I_{AB} = \int (y+r)^2 dA$$

$$I_{AB} = \int (y^2+r^2+2ry)\, dA$$

$$I_{AB} = \int y^2 dA + \int r^2 dA + \int 2ry\, dA$$

By analysing

$\int y^2 dA$ = Moment of inertia about parallel centroidal axis $= I_G$

$\int r^2 dA = r^2 . A$ (about reference axis AB)

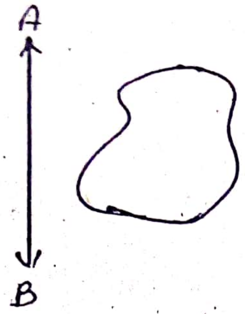
$\int 2ry\, dA = 0$ [as the distance of the centroid from the centroidal axis is "0"]

[∵ the centroidal axis passes through centre of gravity] :

∴ by simplifying

$$\boxed{I_{AB} = I_G + Ar^2}$$

Radius of Gyration:



Radius of gyration of a body is defined as the distance from the reference axis at which the given area is assumed to be compressed and kept as a thin strip, such that there is no change in its moment of inertia

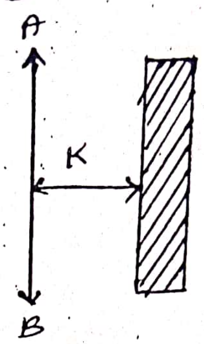


figure (1) shows a plane figure of area 'A'. Let 'I_{AB}' be its moment of inertia about reference axis AB.

Assume the figure to be compressed into a thin strip of the same area 'A' at a distance 'K' from the reference axis AB such that it has same moment of inertia as shown in figure (2)

from the definition

$$\therefore \boxed{I_{AB} = AK^2}$$

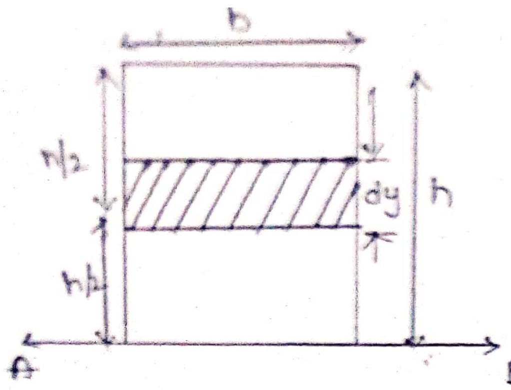
where

$K \rightarrow$ Radius of gyration.

Moment of inertia of some standard general cross section:

1. Rectangle:

Mass of inertia about its base / reference axis



Consider a rectangle of base "b" and height "h". AB is the reference axis through its base, consider an elemental strip of thickness "dy" located at a distance of "y" from the reference axis AB as shown in figure.

Area of elemental strip $dA = b \times dy$

From basic principle $I_{AB} = \int_0^h y^2 \times dA$

$$= \int_0^h y^2 \times b \times dy$$

$$= b\int_0^h y^2 dy = b\left[\frac{y^3}{3}\right]_0^h$$

$$\Rightarrow I_{AB} = \frac{bh^3}{3}$$

Moment of inertia about its centroidal axis:

From parallel axis theorem

$$I_{AB} = I_G + Ar^2$$

$$I_G = I_{AB} - Ar^2$$

$$I_G = \frac{bh^3}{3} - \left[b \times h \times \left(\frac{h}{2}\right)^2\right] \quad \left[\because I_{AB} = \frac{bh^3}{3}\right]$$

here $A = b \times h$, $r = \frac{h}{2}$

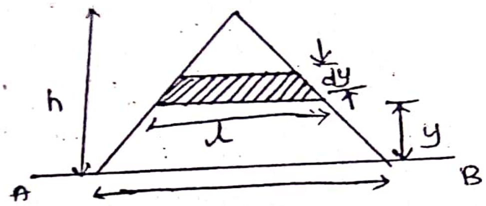
$$I_G = \frac{bh^3}{3} - \frac{bh^3}{4}$$

$$\therefore \boxed{I_G = \frac{bh^3}{12}}$$

2. Triangle:

is the
tal
from

Consider a triangle of base 'b' and height 'h' consider an elemental strip of thickness "dy" which is at a distance of "y" from the reference passes through its base of the triangle



from the similar Δ^{le} principle

$$\frac{l}{b} = \frac{(h-y)}{h}$$

$$l = (h-y)\frac{b}{h}$$

Let dA = Area of the elemental strip

$$dA = l \times dy$$

$$= \frac{b(h-y)}{h} dy$$

from the basic principle $I_{AB} = \int y^2 dA$

$$= \int_0^h y^2 \frac{b}{h}(h-y)\, dy$$

$$= \frac{b}{h}\int_0^h y^2(h-y)\, dy$$

$$= \frac{b}{h}\int_0^h (y^2h - y^3)\, dy$$

$$= \frac{b}{h}\left[\frac{y^3 h}{3} - \frac{y^4}{4}\right]_0^h = \frac{b}{h}\left[\frac{h^4}{12}\right]$$

$$\therefore I_{AB} = \frac{bh^3}{12}$$

Moment of inertia about its centroidal axis

From the parallel axis theorem,

$$I_{AB} = I_G + Ar^2$$

I_G = Moment of inertia about its centroidal axis.

$$\therefore I_G = I_{AB} - Ar^2$$

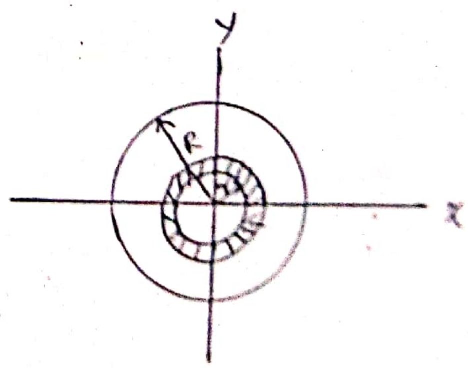
$$= \frac{bh^3}{12} - \left(\frac{1}{2} \times b \times h \times \frac{h^2}{9}\right)$$

$$= \frac{bh^3}{12} - \frac{bh^3}{18}$$

$$\therefore \boxed{I_G = \frac{bh^3}{36}}$$

3. Circular circle

Consider an elemental thin circular ring of thickness dr and radius r shown in figure.



Area of the circular ring $dA = 2\pi r dr$

from the basic principle moment of inertia

$$I_{ZZ} = \int_0^R r^2 dA$$

$$I_{zz} = \int_0 r^2\, 2\pi r\, dr$$

$$= 2\pi \int_0^R r^3 dr \qquad = 2\pi \left[\frac{r^4}{4}\right]_0^R$$

$$= 2\pi \frac{R^4}{4} \qquad = \frac{\pi R^4}{2}$$

$$\therefore \boxed{I_{zz} = \frac{\pi R^4}{2}}$$

from the perpendicular axis theorem, we know that

$$I_{zz} = I_{xx} + I_{yy}$$

But, the given section is symmetrical

$$\therefore I_{xx} = I_{yy}$$

$$I_{zz} = I_{xx} + I_{xx}$$

$$I_{zz} = 2 I_{xx}$$

$$I_{xx} = \frac{I_{zz}}{2} = \frac{\frac{\pi R^4}{2}}{2}$$

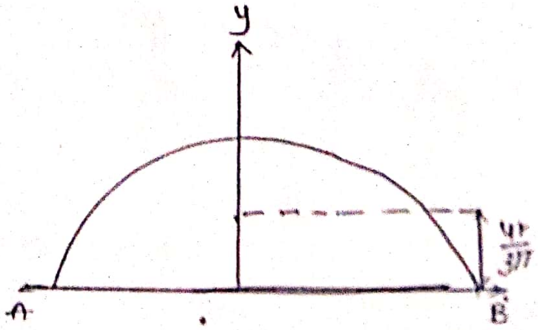
$$\boxed{I_{xx} = \frac{\pi R^4}{4} = I_{yy}}$$

x 4. Semicircle:

Semicircle $= \frac{1}{2} \times$ circle

$$\therefore I_{zz} = \frac{I_{zz}}{2} = \frac{\pi R^4}{4}$$

$$I_{xx} = \frac{\pi R^4}{8} = I_{yy}$$



Moment of inertia about its centroid :

from the parallel axis theorem,

$$I_{AB} = I_G + AR^2$$

$$I_G = I_{AB} - AR^2$$

$$\therefore I_{AB} = I_{xx} = \frac{\pi R^4}{8} ; \quad R = \frac{4R}{3\pi} ; \quad A = \frac{\pi R^2}{2}$$

$$I_G = \frac{\pi R^4}{8} - \frac{\pi R^2}{2}\left[\frac{16R^2}{9\pi^2}\right]$$

$$= \frac{\pi R^4}{8} - \frac{8}{9\pi} \cdot R^4$$

$$= R^4\left(\frac{\pi}{8} - \frac{8}{9\pi}\right)$$

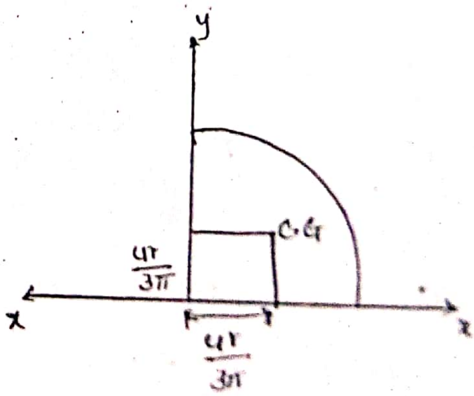
$$\boxed{I_G = 0.116R^4}$$

5. Quarter circle :

Quarter circle $= \frac{1}{4} \times$ circle

$$I_{zz} = \frac{I_{zz}}{4} = \frac{\pi R^4}{8}$$

$$I_{xx} = \frac{\pi R^4}{16} = I_{yy}$$



Moment of inertia about centriodal axis:

from the parallel axis theorem

$$I_{AB} = I_G + AR^2$$

$$I_G = I_{AB} - AR^2$$

$$I_{AB} = I_{xx} = \frac{\pi R^4}{16}, \quad A = \frac{\pi R^2}{4}, \quad \bar{y} = \frac{4R}{3\pi}$$

$$I_G = \frac{\pi R^4}{16} - \left[\frac{\pi R^2}{4} \times \frac{16R^2}{9\pi^2}\right]$$

$$= \frac{\pi R^4}{16} - \frac{4R^4}{9\pi}$$

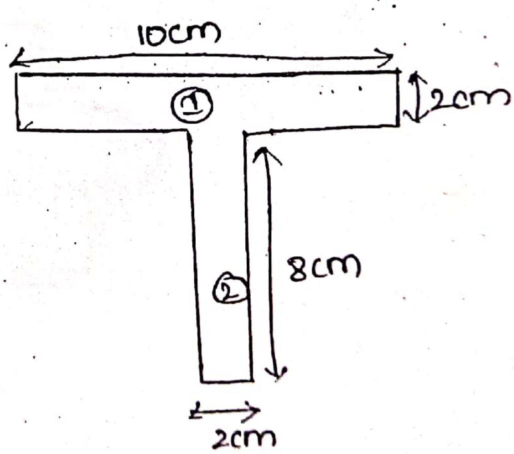
$$= R^4 \left[\frac{\pi}{16} - \frac{4}{9\pi}\right]$$

$$\boxed{I_G = 0.054R^4}$$

· Figure shows a T section of dimensions 10cm x 10cm x 2cm. Determine the moment of inertia of the section about horizontal and vertical axis, passing through the centre of gravity of the section.

A. Given, section is symmetrical

it is divided into 2 parts



1. Rectangle

Area of part 1 $A_1 = 10 \times 2$

$= 20cm^2$

2. Rectangle

Area of part 2 $A_2 = 8 \times 2$

$= 16cm^2$

Total area $A = A_1 + A_2$

$= 20 + 16$

$= 36cm^2$

Let

y_1 = distance of centre of gravity part ① from origin 'O'

$= 8 + \frac{2}{2}$

$= 9cm$

y_2 = distance of centre of gravity part ② from origin 'O'

$= \frac{8}{2} = 4cm$

$\bar{y}$ = distance of centre of gravity from origin 'O'

we know that

$$\bar{y} = \frac{A_1 y_1 + A_2 y_2}{A}$$

$$= \frac{20 \times 9 + 16 \times 4}{36} = \frac{244}{36}$$

$$\boxed{\bar{y} = 6.777 \text{ cm}}$$

⇒ Moment of inertia along xx axis which passing through the centroid of the T section

$$I_{xx} = I_{xx_1} + I_{xx_2}$$

using parallel axis theorem moment of inertia along xx axis passing through its centroid of part ①

$$I_{xx_1} = I_{G_1} + A_1 h_1^2$$

$$I_{G_1} = \frac{bd^3}{12} = \frac{10 \times 2^3}{12} = 6.666 \text{ cm}^4$$

$A_1 = 20 cm^2$

⇒ Mo

its ce

$h_1 = y_1 - \bar{y}$

$= 9 - 6.777 = 2.223\ cm$

$I_{xx_1} = 6.666 + 20 \times (2.223)^2$

$\boxed{I_{xx_1} = 105.411\ cm^4}$

$I_{xx_2} = I_{G_2} + A_2 h_2^2$

$I_{G_2} = \frac{bd^3}{12} = \frac{2 \times 8^3}{12}$

$= 85.333\ cm^4$

$A_2 = 16\ cm^2$

$h_2 = \bar{y} - y_2$

$= 6.777 - 4 = 2.777\ cm$

$I_{xx_2} = 85.333 + 16 \times (2.777)^2 \Rightarrow \boxed{208.72\ cm^4 = I_{xx_2}}$

$\boxed{I_{xx} = 314.136\ cm^2}$

⇒ Moment of inertia along yy axis which passes through its centroid

$I_{yy} = I_{yy_1} + I_{yy_2}$

$I_{yy_1} = \frac{d_1 b_1^3}{12} = \frac{2 \times 10^3}{12}$

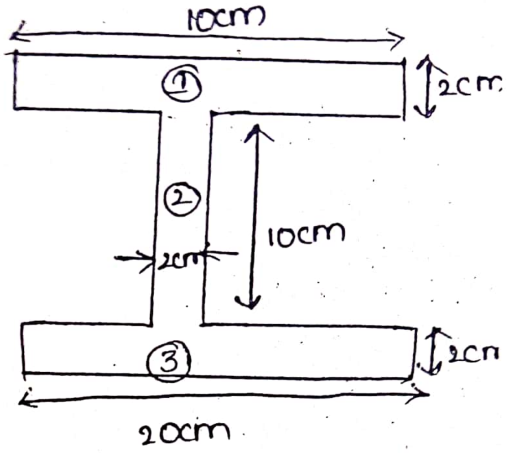
$\boxed{I_{yy_1} = 166.66\ cm^4}$

$I_{yy_2} = \frac{d_2 b_2^3}{12} = \frac{8 \times (2)^3}{12}$

$\boxed{I_{yy_2} = 5.333\ cm^4}$

$$\boxed{I_{yy} = 171.993\ cm^4}$$

2. Find the moment of inertia of the I section shown in fig. about centroidal axis xx perpendicular to the web.



A. Given, section is symmetrical and its divided into 3 parts

1. Rectangle

Area of part ① $A_1 = 10 \times 2$

$= 20cm^2$

2. Rectangle.

Area of part ② $A_2 = 10 \times 2$

$= 20cm^2$

3. Rectangle

Area of part ③ $A_3 = 20 \times 2$

$= 40cm^2$

Total area $A = A_1 + A_2 + A_3$

$A = 80cm^2$

Let

$y_1 = 2 + 10 + \frac{2}{2}$

$= 13cm$

$y_2 = 2 + \frac{10}{2}$

$= 7cm$

$y_3 = \frac{2}{2}$

$= 1cm$

$$\bar{y} = \frac{A_1y_1 + A_2y_2 + A_3y_3}{A}$$

$$= \frac{20\times13 + 20\times7 + 40\times1}{80}$$

$$\boxed{\bar{y} = 5.50cm}$$

⇒ Moment of inertia along xx axis which passing through the centroid of I section

$$\Rightarrow I_{xx} = I_{xx_1} + I_{xx_2} + I_{xx_3}$$

using parallel axis moment of inertia along xx axis

$$I_{xx_1} = I_{G_1} + A_1h_1^2$$

$$I_{G_1} = \frac{10\times(2)^3}{12} = 6.667\ cm^4$$

$$A_1 = 20cm^2$$

$$h_1 = y_1 - \bar{y}$$

$$= 13 - 5.5 = 7.5cm$$

$$I_{xx_1} = 6.667 + 20\times(7.5)^2$$

$$\boxed{I_{xx_1} = 1131.667cm^4}$$

$$I_{xx_2} = I_{G_2} + A_2h_2^2$$

$$I_{G_2} = \frac{2\times(10)^3}{12} = 166.667cm^4 \qquad A_2 = 20cm^2 \qquad h_2 = y_2 - \bar{y}$$

$$= 7 - 5.5$$

$$= 1.5$$

$$I_{xx_2} = 166.667 + 20\times(1.5)^2$$

$$\boxed{I_{xx_2} = 211.667\ cm^4}$$

$$I_{xx_3} = I_{G_3} + A_3 h_3^2$$

$$I_{G_3} = \frac{20 \times 2^3}{12} = 13.333 cm^4 \qquad A_3 = 40 cm^2$$

$$h_3 = \bar{y}_0 - \bar{y}_3 = 5.5 - 1$$

$$= 4.5\ cm$$

$$I_{xx_3} = 13.333 + 40 \times (4.5)^2$$

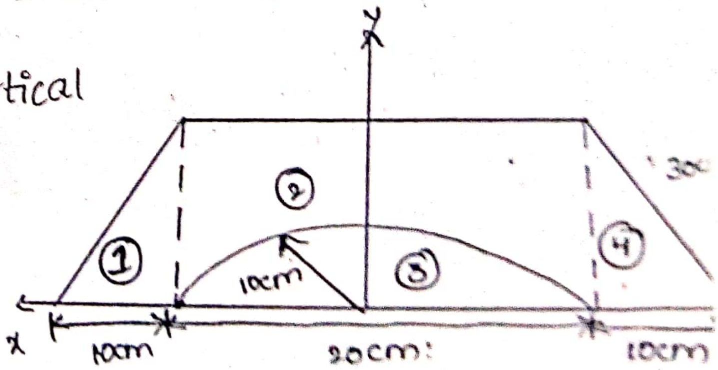
$$\boxed{I_{xx_3} = 823.333 cm^4}$$

$$\Rightarrow I_{xx} = 1131.667 + 211.667 + 823.333$$

$$\boxed{I_{xx} = 2166.667 cm^4}$$

3. Find the moment of inertia of the shaded area shown in fig. about x-axis and y-axis.

A. Given, section is unsymmertical and it divided into 4 parts



1. Right angle triangle

~~Area of part ① A_1 =~~

$$I_{xx_1} = \frac{b_1 h_1^3}{12} = \frac{10 \times (30)^3}{12}$$

$$\boxed{I_{xx_1} = 22500 cm^4}$$

2. Rectangle

$$I_{xx_2} = \frac{b_2 d_2^3}{3}$$

$$= \frac{20 \times (30)^3}{3}$$

$$\boxed{I_{xx_2} = 180000 cm^4}$$

3. Semicircle

$$I_{xx_3} = \frac{\pi R^4}{8} = \frac{\pi \times 10^4}{8}$$

$$\boxed{I_{xx_3} = 3926.990 cm^4}$$

4. Right angle triangle

$$I_{xx_4} = \frac{b_4 h_4^3}{12} = \frac{10 \times (30)^3}{12}$$

$$\boxed{I_{xx_4} = 22500 cm^4}$$

$$\Rightarrow I_{xx} = I_{xx_1} + I_{xx_2} - I_{xx_3} + I_{xx_4}$$

$$\boxed{I_{xx} = 221073.01 cm^4}$$

$\Rightarrow$ Moment of inertia along y-axis

1. $I_{yy_1} = I_{G_1} + A_1 h_1^2$ ($\because$ using parallel axis theorem)

$$I_{G_1} = \frac{hb^3}{36} = \frac{30 \times (10)^3}{36} = 833.333 cm^4$$

$A_1 = \frac{1}{2} \times b \times h = 150 cm^2$ $h_1 = 10 + \frac{10}{3} = 13.333$

$I_{yy_1} = 833.333 + 150 \times (13.333)^2$

$\boxed{I_{yy_1} = 27498.666 cm^4}$

2. $I_{yy_2} = \frac{db^3}{12} = \frac{30 \times (20)^3}{12}$

$\boxed{I_{yy_2} = 20,000 cm^4}$

3. $I_{yy_3} = \frac{\pi R^4}{8} = \frac{\pi \times (10)^4}{8}$

$\boxed{I_{yy_3} = 3926.990 cm^4}$

4. $I_{yy_4} = \frac{hb^3}{36} + A_4 \times h_4^2$

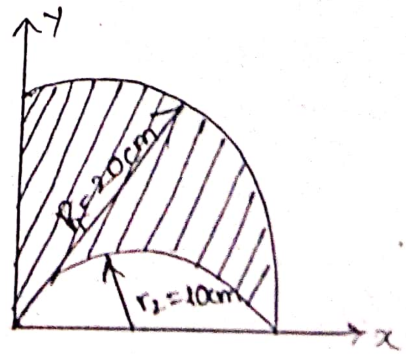
$= \frac{30 \times 10^3}{36} + \frac{1}{2} \times 10 \times 30 \times (13.333)^2$

$\boxed{I_{yy_4} = 27498.666 cm^4}$

$\Rightarrow I_{yy} = I_{yy_1} + I_{yy_2} - I_{yy_3} + I_{yy_4}$

$\boxed{I_{yy} = 71070.342 cm^4}$

4. Find the moment of inertia of shaded area about ox and oy axis also find the centroid of the shaded area.



A. Given section is unsymmetrical it is divided into 2 parts

1. Quater circle

Area of part ① $A_1 = \frac{\pi r_1^2}{4}$

$= \frac{\pi (20)^2}{4} = 314 cm^2$

2. Semicircle

Area of part ② $A_2 = \frac{\pi \times (10)^2}{2}$

$= 157\,cm^2$

Total area $A = A_1 - A_2$

$= 314 - 157$

$= 157\,cm^2$

$x_1 = \frac{4r_1}{3\pi} = \frac{4 \times 20}{3 \times \pi}$ $\qquad$ $x_2 = \frac{20}{2}$

$= 8.49\,cm$ $\qquad$ $\simeq 10\,cm$

$$\bar{x} = \frac{8.49 \times 314 - 157 \times 10}{157}$$

$$\boxed{\bar{x} = 6.98\,cm}$$

$y_1 = \frac{4r_1}{3\pi} = \frac{4 \times (20)}{3 \times \pi}$ $\qquad$ $y_2 = \frac{4r_2}{3\pi} = \frac{4 \times (10)}{3 \times \pi}$

$= 8.49\,cm$ $\qquad$ $= 4.24\,cm$

$$\bar{y} = \frac{8.49 \times 314 - 157 \times 4.24}{157}$$

$$\boxed{\bar{y} = 12.74\,cm}$$

⇒ Moment of inertia about ox axis

$I_{ox} = I_{ox_1} - I_{ox_2}$

$I_{ox_1} = \frac{\pi r_1^4}{16} = 31400$

$$I_{ox_2} = \frac{\pi r_2^4}{16} = 3925$$

$$I_{ox} = 31400 - 3925$$

$$\boxed{I_{ox} = 27475\,cm^4}$$

⇒ Moment of inertia along OY axis

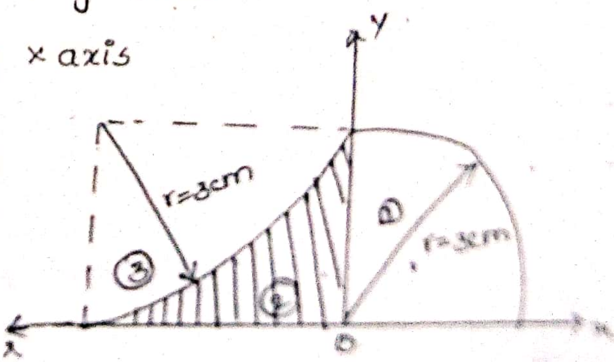
$$I_{oy} = I_{oy_1} - I_{oy_2}$$

$$I_{oy_1} = \frac{\pi r_1^4}{16} = 31400 \qquad I_{oy_2} = \frac{\pi r_2^4}{16} = 3925$$

$$I_{oy} = 31400 - 3925$$

$$\boxed{I_{oy} = 27475\,cm^4}$$

5. Find the moment of inertia of shaded area show in figure about x-axis and y-axis.



A. Moment of inertia along x axis

$$I_{xx} = I_{xx_1} + I_{xx_2} - I_{xx_3}$$

$$I_{xx_1} = \frac{\pi r_1^4}{16} = \frac{\pi \times (3)^4}{16}$$
$$= 15.89\,cm^4$$

$$I_{xx_2} = \frac{bh^3}{3} = 27$$

$$I_{xx_3} = I_{G_3} + A_3 \times r_3^2$$

$$I_{G_3} = 0.054 \times (3)^4 \qquad A_3 = \frac{\pi (3)^2}{4} \qquad r_3 = \left(3 - \frac{4\times 3}{3\pi}\right)^2$$
$$= \qquad\qquad = 14.4183 \qquad\qquad = 63.385$$

$I_{xx} = 15.89 + 27 - 25.504$

$\boxed{I_{xx} = 17.38 \text{ cm}^4}$

$I_{yy} = I_{yy_1} + I_{yy_2} - I_{yy_3}$

$I_{yy_1} = \frac{\pi r_1^4}{16} = 15.89 \text{ cm}^4$

$I_{yy_2} = \frac{bh^3}{3} = 27 \text{cm}^4$

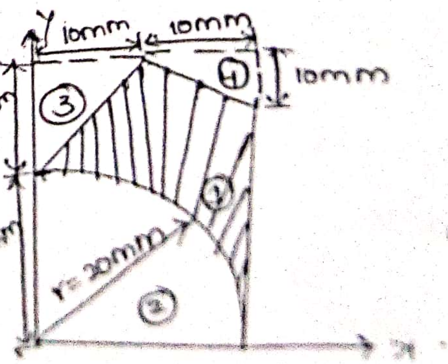
$I_{yy_3} = 0.054 R^4 + \frac{\pi R^2}{4} \times \left(3 - \frac{4\times3}{3\pi}\right)^2$

$= 25.504$

$I_{yy} = 15.89 + 27 - 25.504$

$\boxed{I_{yy} = 17.38 \text{cm}^4}$

6. Find the centroid of the plane area shown in fig. and also determine moment of inertia about its centroidal axis.



A. Let

$\bar{x}$ = distance of C.G from $\bar{o}y$ axis

$\bar{y}$ = distance of C.G from $\bar{o}x$ axis

1. Rectangle

Area $A_1 = 20 \times 35$

$= 700 \text{cm}^2$

2. Quatercircle

Area $A_2 = \frac{\pi}{4} \times (20)^2$

$= 314.158 \text{cm}^2$

3. Right angle triangle

Area $A_3 = \frac{1}{2} \times b \times h$

$= \frac{1}{2} \times 10 \times 15$

$= 75 cm^2$

4. Right angle triangle

Area $A_4 = \frac{1}{2} \times 10 \times 10$

$= 50 cm^2$

Total area $A = A_1 - A_2 - A_3 - A_4$

$= 260.842 \ cm^2$

$x_1 = \frac{20}{2} = 10cm$

$x_2 = \frac{4r}{3\pi} = \frac{4 \times 20}{3 \times 3.141} = 8.488cm$

$x_3 = \frac{10}{3} = 3.333cm$

$x_4 = 10 + \frac{12}{3} \times 10 = 16.666cm$

$$\bar{x} = \frac{10 \times 700 - 8.488 \times 314.158 - 75 \times 3.333 - 50 \times 16.66}{260.842 cm^2}$$

$\boxed{\bar{x} = 12.460cm}$

$y_1 = \frac{35}{2} = 17.5cm$

$y_2 = \frac{4 \times 20}{3 \times 3.141} = 8.488cm$

$y_3 = 20 + \frac{2}{3} \times 15 = 30cm$

$y_4 = 25 + \frac{2}{3} \times 10 = 31.666cm$

$$\bar{y} = \frac{17.5 \times 700 - 8.488 \times 314.158 - 30 \times 3.333 - 50 \times 31.666}{260.842 \ cm^2}$$

$\boxed{\bar{y} = 22.044cm}$

⇒ Moment of inertia about xx axis

$$I_{xx} = I_{xx_1} - I_{xx_2} - I_{xx_3} - I_{xx_4}$$

$$I_{xx_1} = \frac{bd^3}{3} = \frac{20\times(35)^3}{3}$$

$$\boxed{I_{xx_1} = 285833.333 cm^4}$$

$$I_{xx_2} = \frac{\pi R^4}{16} = \frac{\pi\times(20)^4}{16}$$

$$\boxed{I_{xx_2} = 31415.926 cm^4}$$

$$I_{xx_3} = I_{G_3} + A_3 \times h_3^2$$

$$I_{G_3} = \frac{bh^3}{36} = \frac{10\times(15)^3}{36}$$

$$= 937.5 cm^4$$

$A_3 = 75 cm^2$

$h_3 = 35 - \frac{15}{3}$

$= 30 cm.$

$$I_{xx_3} = 937.5 + 75\times(30)^2$$

$$\boxed{I_{xx_3} = 68437.5 cm^4}$$

$$I_{xx_4} = I_{G_4} + A_4 \times h_4^2$$

$$I_{G_4} = \frac{bh^3}{36} = \frac{10\times(10)^3}{36}$$

$$= 277.777 cm^4$$

$A_4 = 50 cm^2$.

$h_4 = 35 - \frac{10}{3}$

$= 31.666 cm$

$$\boxed{I_{xx_4} = 50414.554 cm^4}$$

$$I_{xx} = 285833.333 - 31415.926 - 68437.5 - 50414.554$$

$$\boxed{I_{xx} = 135565.353 cm^4}$$

⇒ Moment of inertia about YY axis

$$I_{yy_1} = \frac{db^3}{3} = \frac{35\times(20)^3}{3} = 93333.333 cm^4$$

$I_{yy_2} = \frac{\pi R^4}{16} = \frac{\pi \times (20)^4}{16}$

$= 31415.926 \text{ cm}^4$

$I_{yy_3} = \frac{bh^3}{12} = \frac{10 \times (15)^3}{12}$

$= 1250$

$I_{yy_4} = I_G + A_4 r_4^2$

$I_{G_4} = \frac{10 \times (10)^3}{36}$

$= 277.777 \text{ cm}^4$

$A_4 = 50 \text{ cm}^2$ $\quad h_4 = 20 - \frac{10}{3}$

$= 16.666$

$I_{yy_4} = 14172.145$

$I_{yy} = 93333.333 - 31415.926 - 1250 - 14172.145$

$\boxed{I_{yy} = 46501.85 \text{ cm}^4}$

$I_{xx} = I_{Gx} + A_x \cdot \bar{y}^2$

$I_{Gx} = I_{xx} - A\bar{y}^2$

$= 135565.352 - 260.841 \times (22.045)^2$

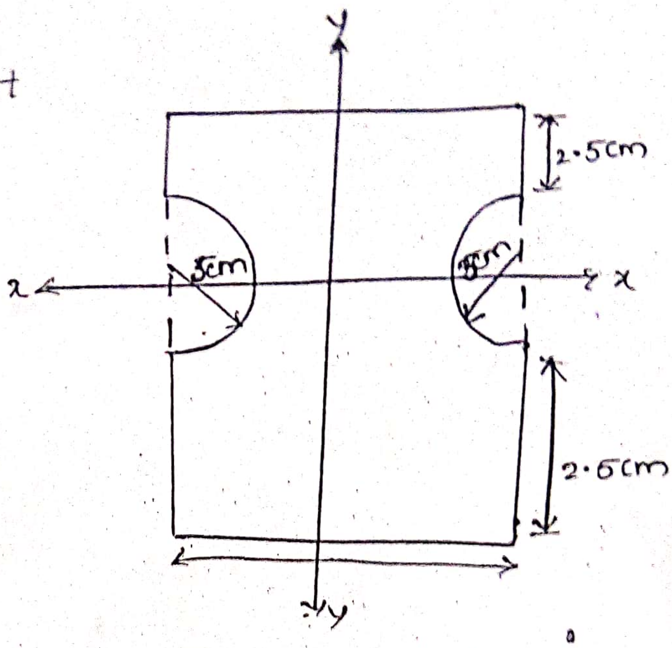
$\boxed{I_{Gx} = 8801.314 \text{ cm}^4}$

$I_{yy} = I_{Gy} + A \cdot \bar{x}^2$

$I_{Gy} = 46501.853 + (-260.841)(12.460)^2$

$\boxed{I_{Gy} = 6003.810 \text{ cm}^4}$

5. Determine the moment of inertia of the plane area shown in fig about the centroidal x axis and centroidal y axis.



1. Moment of inertia about centroidal x axis

$$I_{G_x} = I_{Gx_1} - I_{Gx_2} - I_{Gx_3}$$

$$I_{Gx_1} = \frac{bh^3}{12} = \frac{12(15)^3}{12}$$

$$= 3375 \text{ cm}^4$$

$$I_{Gx_2} = \frac{\pi r^4}{8} = \frac{\pi (5)^4}{8}$$

$$= 245.436 \text{ cm}^4$$

$$I_{Gx_3} = \frac{\pi r^4}{8} = \frac{\pi (5)^4}{8}$$

$$= 245.436 \text{ cm}^4$$

$$I_{Gx} = 3375 - 245.436 - 245.436$$

$$\boxed{I_{Gx} = 2884.375 \text{ cm}^4}$$

⇒ Moment of inertia about y axis

$$I_{Gy} = I_{Gy_1} - I_{Gy_2} - I_{Gy_3}$$

$$I_{Gy_1} = \frac{bh^3}{12} = \frac{15(12)^3}{12}$$

$$\cancel{= 3375 \text{ cm}^4} \quad = 2160 \text{ cm}^4$$

$I_{Gy_2} = 0.11R^4 + Ar^2$

$= 0.11R^4 + \frac{\pi(5)^2}{2} \times \left(6 - \frac{4\times5}{3\pi}\right)^2$

$= 0.11 \times (5)^4 + 68.75 \times 15.038$

$= 659.291 \text{ cm}^4$

$I_{Gy_3} = 0.11R^4 + Ar^2$

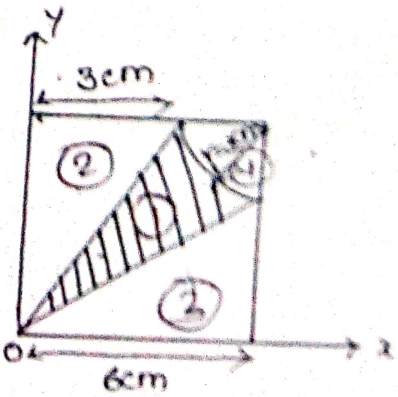
$= 0.11(5)^4 + \frac{\pi(5)^2}{2} \times \left(6 - \frac{4\times5}{3\pi}\right)^2$

$= 659.291 \text{ cm}^4$

$I_{Gy} = 2160 - 659.291 - 659.291 \text{ cm}^4$

$\boxed{I_{Gy} = 841.39 \text{ cm}^4}$

8. Determine the co-ordinates of centroid of shaded area shown in fig. Also determine the moment of inertia at base x-axis and transfer it to the centroid x axis using parallel axis theorem.



A. 1. Square

Area of part ① $A_1 = 6 \times 6$

$= 36 \text{ cm}^2$

2. Right angle triangle

Area of part ② $A_2 = \frac{1}{2} \times 6 \times 3$

$= 9 \text{ m}^2$

3. Right angle triangle

Area of part ③ $A_3 = \frac{1}{2} \times 3 \times 6.$

$= 9 m^2$

4. Semicircle

Area of part ④ $A_4 = \frac{\pi (3)^2}{4}$

$= 7.065 m^2$

Total area $A = A_1 - A_2 - A_3 - A_4$

$A = 10.935 m^2$

$x_1 = \frac{6}{2} = 3m$ $\qquad x_2 = \frac{2b}{3} = \frac{2(6)}{3} = 4m$

$x_3 = \frac{b}{3} = \frac{3}{3} = 1m$ $\qquad x_4 = 6 - \frac{4r}{3\pi} = 6 - \frac{4(3)}{3\pi} = 4.726m$

ded

inertic

axis

$\bar{x} = \frac{3 \times 36 - 4 \times 9 - 1 \times 9 - 7.065 \times 4.726}{10.935}$

$\boxed{\bar{x} = 2.707 m}$

$y_1 = \frac{6}{2} = 3m$ $\qquad y_2 = \frac{h}{3} = \frac{3}{3} = 1m$

$\rightarrow x$

$y_3 = \frac{2}{3} \times h = \frac{2}{3} \times 6$ $\qquad y_4 = 6 - \frac{4r}{3\pi} = 6 - \frac{4 \times (3)}{3\pi}$

$= 4m$

$= 4.726m$

$$\bar{y} = \frac{3\times 36 - 1\times 9 - 4\times 9 - 4.276\times 7.065}{10.935}$$

$$\boxed{\bar{y} = 2.707\ m}$$

⇒ Moment of inertia about its base x axis

$$I_{xx} = I_{xx_1} - I_{xx_2} - I_{xx_3} - I_{xx_4}$$

$$I_{xx_1} = \frac{bh^3}{3} = \frac{(6)(6)^3}{3}$$

$$= 432\ m^4$$

$$I_{xx_2} = \frac{bh^3}{12} = \frac{6\times(3)^3}{12}$$

$$= 13.5\ m^4$$

$$I_{xx_3} = \frac{bh^3}{6} + 9\times\left(6-\frac{6}{3}\right)$$

$$= \frac{3\times(6)^3}{6} + 9\times\left(6-\frac{6}{3}\right)$$

$$= 162\ m^4$$

$$I_{xx_4} = I_{G_4} + A_4 y_4^2 = 0.055\times(3)^4 + \frac{\pi\times(3)^2}{4}\times\left(6-\frac{4\times(3)}{3\pi}\right)$$

$$= 4.455 + 7.065\times 22.372$$

$$= 162.519$$

$$I_{xx} = 432 - 13.5 - 162 - 162.519$$

$$\boxed{I_{xx} = 93.38}$$

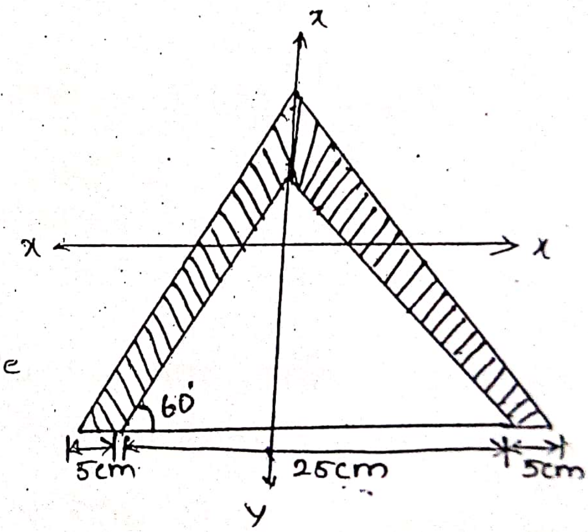
$$\Rightarrow I_{xx} = I_{Gx} + Ar^2$$

$$I_{Gx} = I_{xx} - Ar^2$$

$$= 9.98 - 10.935(2.707)^2$$

$$\boxed{\therefore I_{Gx} = 13.89\ m^4}$$

7. Figure shows a plane area find the moment of inertia of the section about xx axis and yy axis passing through the centre of gravity of this section.



1. Height of the larger Δ^{le}

$$\tan 60° = \frac{H}{17.5}$$

$$\Rightarrow H = 30.31\ cm$$

Similarlly height of smaller Δ^{le}

$$\tan 60° = \frac{h}{12.5}$$

$$\Rightarrow h = 21.65\ cm$$

$\frac{4\times(3)}{3\pi}$

1. Right angle triangle

Area of part ① $A_1 = \frac{1}{2} \times 35 \times 30.31$

$= 530.425\ cm^2$

2. Right angle triangle

Area of part ② $A_2 = \frac{1}{2} \times 25 \times 21.65$

$= 270.625\ cm^2$

Total area $A = A_1 - A_2$

$$A = 259.88 \text{ cm}^2$$

$$y_1 = \frac{30.31}{3} = 10.103 \text{cm} \qquad y_2 = \frac{21.65}{3} = 7.216 \text{ cm}$$

$$\bar{y} = \frac{10.103 \times 530.425 - 7.216 \times 270.625}{259.88}$$

$$\boxed{\bar{y} = 13.1102 \text{cm}}$$

⇒ Moment of inertia along yy axis

$$I_{yy} = I_{yy_1} - I_{yy_2}$$

$$I_{yy_1} = 2 \times \frac{bh^3}{12} \qquad I_{yy_2} = 2 \times \frac{bh^3}{12}$$

$$= 2 \times \frac{(30.31)(17.5)^3}{12} \qquad = 2 \times \frac{21.65 \times (12.5)^3}{12}$$

$$= 27073.776 \qquad = 7.04752$$

$$I_{yy} = 27073.776 - 7.04752$$

$$\boxed{I_{yy} = 20026.256 \text{cm}^4}$$

⇒ from parallel axis theorem

$$I_{AB} = I_{Gx} + Ar^2$$

$$I_{Gx} = I_{AB} - Ar^2$$

$$= 60075.233 - 259.865 \times (13.11)^2$$

$$= 60075.23 - 44653.733$$

$$\boxed{I_{Gx} = 15421.497\, cm^4}$$

Q. For the given shaded area shown in fig. find

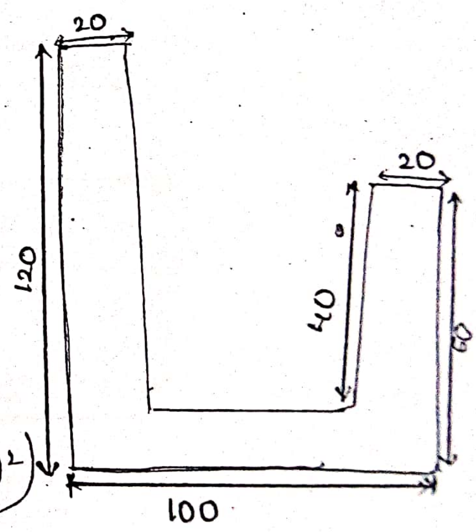
i, M.I about the reference axis

ii, M.I about the centrodial axis

iii, Polar moment of inertia about the origin O

iv, Radius of gyration about reference axis

v, Radius of gyration about centrodial axis



i, M.I about the reference axis

$$I_{ox} = \frac{20\times(120)^3}{3} + \frac{60\times(20)^3}{3} + \frac{20\times(60)^3}{3}$$

$2.5)^3$

$$\therefore \boxed{I_{ox} = 13120000}$$

$$I_{oy} = \frac{120\times(20)^3}{3} + \left(\frac{20\times(60)^3}{12} + 20\times 60\times(50)^2\right)$$

$$+ \left(\frac{60\times(20)^3}{12} + 60\times 20 \times (90)^2\right)$$

$$= 320000 + 3360000 + 3760000$$

$$\boxed{\therefore I_{oy} = 13440000}$$

ii, Moment of inertia about centrodial axis

$\Rightarrow$ from parallel axis theorem

Area of part ① $A_1 = 120 \times 20$

$= 2400 mm^2$

Area of part ② $A_2 = 60 \times 20$

$= 1200 mm^2$

Area of part ③ $A_3 = 20 \times 60$

$= 1200 mm^2$

Total area $A = 4800 mm^2$

$$\bar{x} = \frac{(120 \times 20 \times 10) + (60 \times 20 \times 50) + (20 \times 60 \times 90)}{4800}$$

$$\bar{x} = \frac{192000.000}{4800}$$

$$\boxed{\bar{x} = 40 mm}$$

$$\bar{y} = \frac{(120 \times 20 \times 60) + (60 \times 20 \times 10) + (20 \times 60 \times 30)}{4800}$$

$$\bar{y} = \frac{192000.000}{4800}$$

$$\boxed{\bar{y} = 40 mm}$$

By parallel axis theorem, we have

$$I_{Gx} = I_{xxG} + A\bar{y}^2$$

$$I_{xxG} = 13120000 - (4800 \times 40^2)$$

$$\boxed{I_{xxG} = 5440000 mm^2}$$

$$I_{ox} = I_{yyG} + A\bar{x}^2$$

$$I_{yyG} = 13440000 - (4800 \times 40^2)$$

$$\boxed{I_{yyG} = 5760000\,mm^2}$$

iii, Polar moment of inertia about origin O

$$I_p = I_{ox} + I_{oy}$$

$$= 13120000 + 13440000$$

$$\boxed{I_p = 26560000\,mm^4}$$

iv, Radius of gyration about reference axis

$$I_{ox} = \sqrt{\frac{I_{ox}}{A}} = \sqrt{\frac{13120000}{4800}}$$

$$\boxed{I_{ox} = 52.28\,mm}$$

$$I_{oy} = \sqrt{\frac{I_{oy}}{A}} = \sqrt{\frac{13140000}{4800}}$$

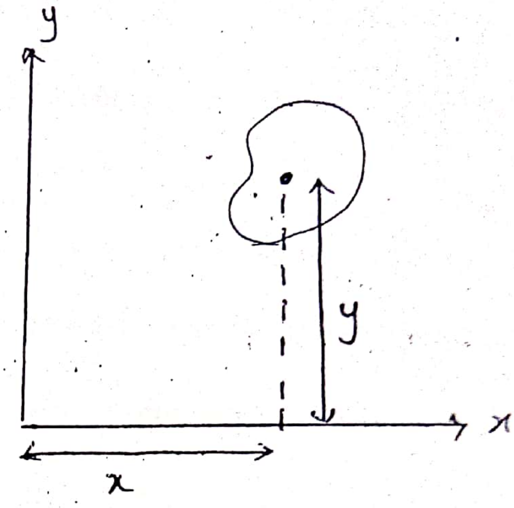
$$\boxed{I_{oy} = 52.92\,mm}$$

v, Radius of gyration about centrodial axis

$$K_{xxG} = \sqrt{\frac{I_{xxG}}{A}} = 33.67\,mm$$

$$k_{yyG} = \sqrt{\frac{I_{yyG}}{A}} = 34.64\,mm$$

Mass moment of inertia:



- Consider a body of mass 'm' shown in figure.

Let x = distance of C.G of mass (m) from oy axis

y = distance of C.G of mass 'm' from ox axis

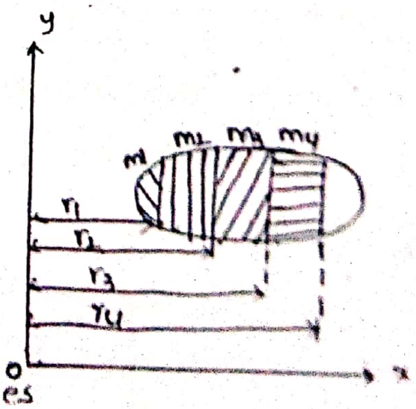
The first moment of mass $= M \times x$ (along OY axis)

The first moment of mass $= M \times y$ (along ox axis)

Second moment of mass $= M \times x^2$ (along oy axis)

Second moment of mass $= M \times y^2$ (along ox axis)

Hence, the product of mass and the square of the distance of the centre of gravity of the mass from an axis is known as Mass moment of inertia about that axis. It is denoted by "Im".



Consider a body which is split up into no. of small masses m_1, m_2, m_3 ----- Let the centre of gravity of the masses from a given axis be at a distances r_1, r_2, r_3 ----

$\therefore$ mass moment of inertia $I_m = m_1 r_1^2 + m_2 r_2^2 + \cdots$

The above equation can be written as

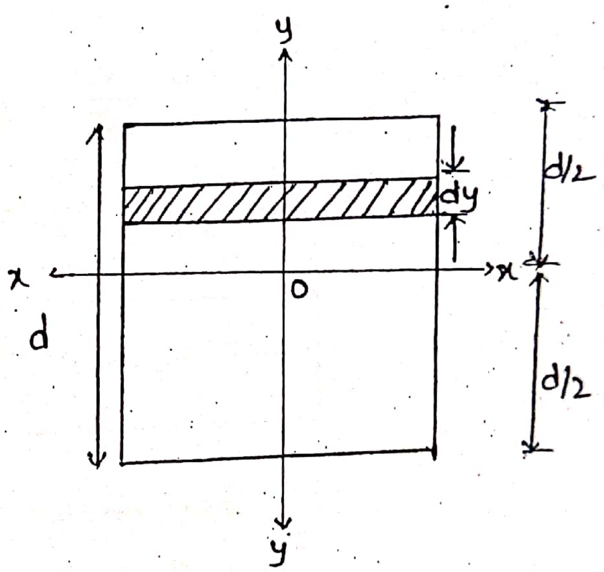
$$I_m = \Sigma m_i r_i^2 \quad (\text{where } i = 1, 2, 3 \ldots)$$

If the masses are in large number

$$\therefore \boxed{I_m = \int dM \times r^2}$$

Mass moment of inertia for different sections:

1. Thin rectangular plate:



Consider a rectangular plate of thickness "t" and width "b" & "d"

Consider a small element of depth "dy" and at a distance "y" from xx axis

Area of the elemental strip $= b \times dy$

Volume of the elemental strip $= b \times t \times dy$

Mass of the element dm = density × volume

$$dm = \rho \times bt \times dy$$

Mass moment of inertia of an element $(I_{dm})_{xx}$

$$= dm \times y^2$$

$$= \rho\, dy\, bt \times y^2$$

$$\Rightarrow (I_{dm})_{xx} = \rho\, bt\, y^2 \times dy$$

The total mass moment of inertia is obtained by integrating between the limits $-d/2$ to $d/2$

$$\Rightarrow (I_m)_{xx} = \int_{-d/2}^{d/2} (I_{dm})_{xx}$$

$$= \int_{-d/2}^{d/2} \rho\, dy\, bt y^2 = \rho bt \int_{-d/2}^{d/2} y^2\, dy$$

$$= \rho bt \left(\frac{y^3}{3}\right)_{-d/2}^{d/2} = \frac{\rho bt}{3}\left[\frac{d^3}{8} + \frac{d^3}{8}\right]$$

$$= \frac{\rho bt}{3} \times \frac{d^3}{4} = \frac{\rho bt d^3}{12}$$

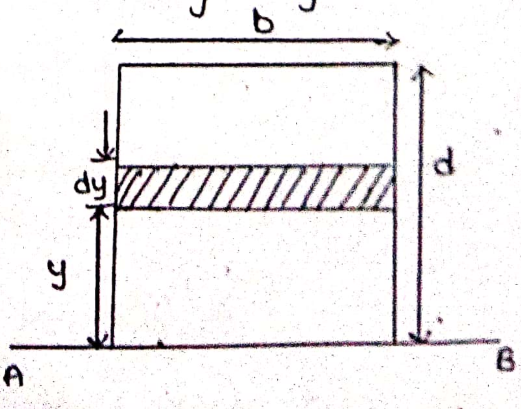
$$(I_m)_{xx} = \frac{\rho bt \times d \times d^2}{12}$$

$$\boxed{(I_m)_{xx} = \frac{Md^2}{12}} \qquad (\text{where } M = \rho btd)$$

Similarlly the mass moment of inertia of thin rectangular plate about yy axis passing through c.G of the plate is given by

$$\boxed{(I_m)_{yy} = \frac{Mb^2}{12}}$$

Mass moment of inertia of a rectangular plane about a line passing through its base



Consider an elemental strip of thickness "dy" which is at a distance "y" from a line AB passing through the base of the rectangular plate of

about

B

Area of elemental strip $= b \times dy$

Volume of elemental strip $= b \times t \times dy$

mass of element dm = density $\times$ volume

$= \rho bt \times dy$

Mass moment of inertia is obtained by integrating between the limits of 0 to d

Mass moment of inertia of an element

$$(I\,dm)_{xx} = dm \times y^2$$

$$(I\,dm)_{xx} = \rho bt\, y^2\, dy$$

$$\Rightarrow (I_{m\,xx}) = \int_0^d (I\,dm)_{xx}$$

$$= \int_0^d \rho bt\, y^2\, dy$$

$$= \rho bt \left[\frac{y^3}{3}\right]_0^d = \frac{\rho bt \cdot d^3}{3}$$

$$\boxed{(I_{mxx}) = \frac{Md^2}{3}}$$

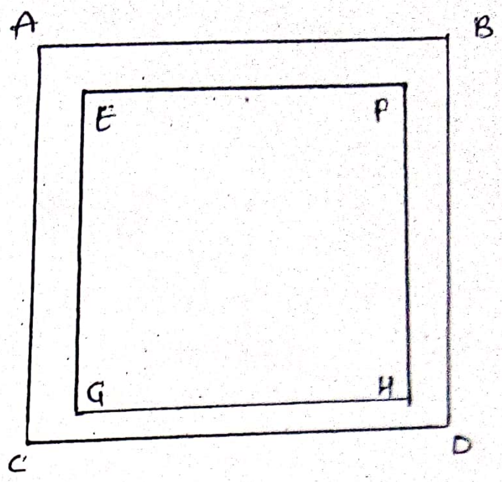
where $M = \rho bt\, d$

Similarlly the mass moment of inertia of thin rectangular plate with a line AB passing through its Base along y axis is given by

$$\boxed{(I_m)_{yy} = \frac{Mb^2}{3}}$$

Mass moment of inertia of a hallow rectangular plate:

Figure shows a hallow rectangular plate in which ABCD is the main plate & EFGH is the cut out section plate



The mass moment of inertia of the main plate ABCD about xx axis is given by

$$(I_m xx) = \frac{Md^2}{12}$$

The mass moment of inertia of the cut out section EFGH about xx axis $= \frac{m d_1^2}{12}$ $[\because m = \rho b_1 t_1 \times d_1]$

where M = mass of the main plate

m = mass of the cutout section

$\therefore$ The mass moment of inertia of hallow rectangular plate

$$\boxed{(I_m)xx = \frac{Md^2}{12} - \frac{md_1^2}{12}}$$

2. Thin circular plate:

Consider a circular plate of radius "R" and thickness t with "O" as the centre

Consider an elemental circular ring of radius "r" and width "dr" as shown in fig.

Area of the circular ring $= 2\pi r\,dr = dA$

volume of the elementary circular ring $= dA \times t$

$= 2\pi r t \times dr$

Mass of the circular elementary ring

$$dm = \text{density} \times \text{volume}$$

$$dm = \rho\, 2\pi r t \times dr$$

Mass moment of inertia of elementary circular ring

$$(I_{dm})_{zz} = dm \times r^2$$

$$= \rho\, 2\pi r^3 t \times dr$$

The total mass moment of inertia of a circular plate is obtained by integrating between the limits 0 to R

$$(I_m)_{zz} = \int_0^R \rho\, 2\pi r^3 t\, dr$$

$$= \int_0^R \rho\, 2\pi r^3 t\, dr \quad = \rho\, 2\pi t \left[\frac{r^4}{4}\right]_0^R$$

$$= 2\pi \rho t \times \frac{R^4}{4}$$

$$= \rho \pi R^2 \times \frac{R^2}{2}$$

$$= \frac{\rho V R^2}{2}$$

$$\boxed{(I_m)_{zz} = \frac{MR^2}{2}}$$

The given section is symmetrical,

$$I_{xx} = I_{yy}$$

from perpendicular axis theorem

$$I_{zz} = I_{xx} + I_{yy}$$

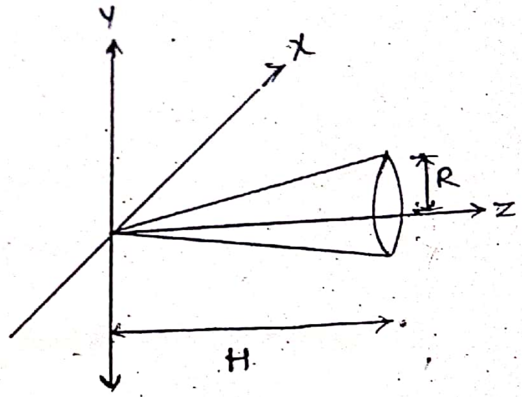
$$I_{zz} = 2 I_{xx} \qquad (\because I_{xx} = I_{yy})$$

$$I_{xx} = \frac{I_{zz}}{2}$$

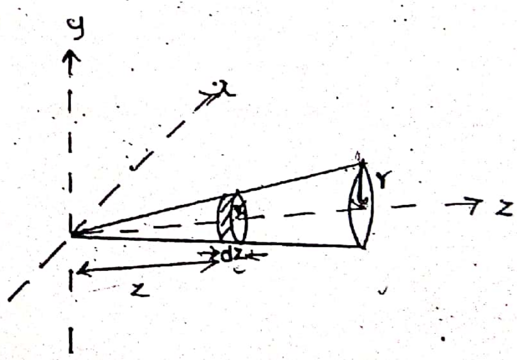
$$\boxed{I_{xx} = \frac{MR^2}{4}}$$

3. Right circular cone :

Consider a cone of base radius "R", height "H" and mass, density "ρ" oriented with respect to axis as shown in the figure.



Suppose we cut a circular disc of radius "r" and thickness "dz" at a distance "z" from the origin. Then its mass is given by



$$dm = \rho \times \pi r^2 \times dz$$

The mass moment of inertia is given by $(dI_m)_{zz}$

$$\therefore (dI_m)_{zz} = dm \times \frac{r^2}{2}$$

$$(I_{dm})_{zz} = \frac{\rho \pi r^2 . r^2 dz}{2}$$

on integrating between the limits 0 to H we get

Mass moment of inertia of the cone is obtained as $\quad (I_m)_{zz} = \int_0^H \rho \pi r^2 \frac{r^2}{2} dz$

from similar Δ'les

$$\frac{r}{R} = \frac{z}{H}$$

$$r = \frac{z}{H} \times R$$

$$(I_m)_{zz} = \int_0^H \rho\pi \frac{r^4}{2} dz$$

$$= \frac{\rho\pi}{2} \int_0^H \left(\frac{z}{H} \times R\right)^4 dz = \frac{\rho\pi}{2} \times \frac{R^4}{H^4} \int_0^H z^4 dz$$

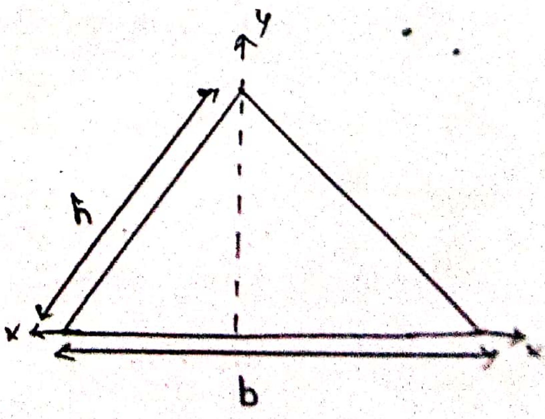
$$= \frac{\rho\pi}{2} \cdot \frac{R^4}{\cancel{H^4}} \cdot \frac{H^{\cancel{5}}}{5}$$

$$(I_m)_{zz} = \frac{\rho\pi R^4 H}{10}$$

$$\therefore \boxed{(I_m)_{zz} = \frac{3}{10} \times M \times R^2}$$ $[\because V = \frac{1}{3} \times \pi r^2 \times H$ & $\rho V = M]$

4. Triangular plate :

Consider a thin Δ'lar plate of base "b" height "h" & thickness "t"

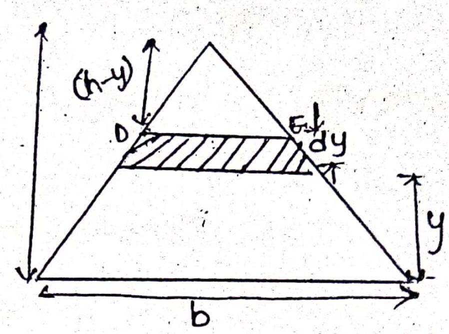


Consider a thin horizontal strip "DE" of thickness "dy" at a distance "y" from the base

from similar Δ'le principle

$$\frac{DE}{b} = \frac{h-y}{h}$$

$$DE = (h-y)\frac{b}{h}$$



Area of the thin strip,

$$dA = DE \times dy$$

$$= \frac{b}{h}(h-y)\, dy$$

volume of rectangular strip $dv = dA \times$ thickness of strip

$$= \frac{b}{h}(h-y) \times t \times dy$$

Mass moment of inertia of the elemental strip (dI_m),

$$\therefore\ dm = \rho \times dv$$

$$(dI_m)_{xx} = dm \times y^2$$

$$= \rho \cdot \frac{b}{h}(h-y) \times t \times dy \times y^2$$

By integrating the above equation between the limits 0 to h to obtain total mass moment of inertia along xx axis

$$(I_m)_{xx} = \int_0^h \rho \frac{b}{h}(h-y)\, y^2 \times t \times dy$$

$$= \rho \frac{b}{h} t \int_0^h hy^2 - y^3 dy = \frac{\rho bt}{h}\left[\frac{hy^3}{3} - \frac{y^4}{4}\right]_0^h$$

$$= \frac{\rho bt}{h} \cdot \frac{h^4}{12}$$

$$(I_m)_{xx} = \frac{\rho bth^3}{12} = \rho \times \frac{1}{2} \times b \times h \times \frac{h^2}{6} \times t$$

$$\boxed{(I_m)_{xx} = \frac{Mh^2}{6}}$$

$$\left[\because V = \frac{1}{2} \times b \times h \times t \quad \rho V = M\right]$$

from parallel axis theorem,

$$(I_m)_{xx} = (I_{gm})_{xx} + M \cdot r^2$$

$(I_{cm})_{xx} = (I_m)_{xx} - Mr^2$

$= \frac{Mh^2}{6} - M \times \frac{h^2}{9}$ $\left[\because r = \frac{h}{3}\right]$

$$\boxed{(I_{cm})_{xx} = \frac{Mh^2}{18}}$$